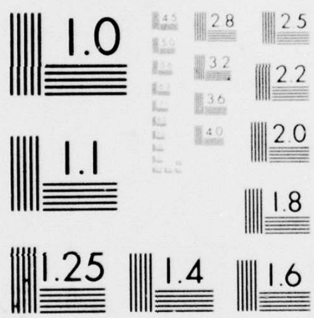
AD-A038 785 INDIANA UNIV BLOOMINGTON DEPT OF MATHEMATICS F/G 12/1
A ROBUST SPHERICAL CORRELATION COEFFICIENT AGAINST SCALE.(U)
1977 K V MARDIA, M L PURI AF-AFOSR-2927-76
UNCLASSIFIED AFOSR-TR-77-0447 NL

1 OF 1
AD
A038785

END
DATE
FILMED
5-77



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

A Robust Spherical Correlation Coefficient Against Scale

By Kanti V. Mardia and Madan L. Puri

University of Leeds, U.K., and Indiana University, Bloomington

SUMMARY

There are several bi-directional situations where it is required to obtain a measure of correlation. The question has been raised often that the bi-directional correlation coefficients so far known are not scale invariant even asymptotically. Following Cox's procedure (Cox, 1975), we introduce a new correlation coefficient which has the desirable property of being invariant under scale for large samples with von Mises marginals. We obtain its asymptotic distribution under the hypothesis of independence. We examine its properties, and give a numerical example.

Work supported by the Air Force Office of Scientific Research, AFSC, USAF, under Grant No. AFOSR76-2927. Reproduction in whole or in part is permitted for any purpose of the U.S. Government.

1. Cox's Procedure

Cox (1975) derived tests for "von Miseness" for circular observations using the exponential distribution with probability density function proportional to

$$\exp(\alpha_1 \cos x + \beta_1 \sin x + \alpha_2 \cos 2x + \beta_2 \sin 2x) .$$

If $X_1, \ldots, X_n$ is a random sample, then the agreement with the von Mises density $\alpha_2 = \beta_2 = 0$ is thus tested from the conditional distribution of $(\Sigma \cos 2X_j , \Sigma \sin 2X_j)$ given $(\Sigma \cos X_j, \Sigma \sin X_j)$.

Let $(X_1,Y_1),\ldots,(X_n,Y_n)$ be n independent observations on a torus. Mardia (1975) introduced the correlation coefficient of (X_i,Y_i) , $i=1,\ldots,n$ as

$$\max(D_+,D_-)/\{(1-\bar{R}_1^2)(1-\bar{R}_2^2)\} , \qquad (1.1)$$

where

$$n^2 D_\pm = \{\sum_{i=1}^{n} \cos(X_i^* \pm Y_i^*) - n\bar{R}_1\bar{R}_2\}^2 + \{\sum_{i=1}^{n} \sin(X_i^* \pm Y_i^*)\}^2 ,$$

$X_i^* = (X_i - \bar{X}_0) \mod 2\pi$ and $Y_i^* = (Y_i - \bar{Y}_0) \mod 2\pi$, and where $\bar{R}_1$ and $\bar{R}_2$ are the mean resultant lengths and $\bar{X}_0$, $\bar{Y}_0$ are the mean directions of X_i and Y_i respectively. (Downs and Eifler (1975) show the equivalence of this correlation with a correlation coefficient introduced by Downs (1974) in a different set up). Downs and Eifler (1975) point out that the distribution of (1.1) will in general depend on the concentration parameter of the marginals. (For a summary of various circular correlation coefficients, see Puri and Rao (1976)).

It is shown that Cox's procedure (Cox, 1975) provides a new correlation coefficient which is Scale invariant for large samples with von Mises marginals.

Consider the probability density function introduced by Mardia (1975) as

Kappa P.C.

$$C \exp\{\hat{\kappa}_1 \cos(x-\mu) + \kappa_2 \cos(y-\nu) + a \cos x \cos y + b \sin x \cos y$$
$$+ c \cos x \sin y + d \sin x \sin y\} \ , \quad 0 \leq x, \ y \leq 2\pi \ .$$

Following Cox's procedure, to test the hypothesis of independence (i.e. $a = b = c = d = 0$) with von Mises marginals, we should consider the distribution of

$$n\bar{\underset{\sim}{U}}' = (\Sigma \cos X_i \cos Y_i, \ \Sigma \cos X_i \sin Y_i, \ \Sigma \sin X_i \cos Y_i, \ \Sigma \sin X_i \sin Y_i$$

given

$$n\bar{\underset{\sim}{T}}' = (\Sigma \cos X_i, \ \Sigma \sin X_i, \ \Sigma \cos Y_i, \ \Sigma \sin Y_i)$$

To obtain a correlation coefficient, a suitable function of $\bar{\underset{\sim}{U}} | \bar{\underset{\sim}{T}}$ is considered in section 2. Its asymptotic distribution under the null hypothesis is considered in section 3. The properties of the resulting coefficient is examined in section 4 together with its spherical extension. In section 5, a numerical example is given.

2. An Invariant Function

Consider the hypothesis H of independence: $a = b = c = d = 0$. Under H , let

$$\mathrm{Cov}(\bar{\boldsymbol{U}},\bar{\boldsymbol{T}}) = \frac{1}{n}\boldsymbol{\Sigma} = \frac{1}{n}\begin{pmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12}\\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{pmatrix} .$$

Set

$$\boldsymbol{\Sigma}_{1.2} = \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}$$

and denote

$$g(\boldsymbol{X},\boldsymbol{Y}) = [\bar{\boldsymbol{U}} - E_H\bar{\boldsymbol{U}} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\bar{\boldsymbol{T}} - E_H\bar{\boldsymbol{T}})]'\,\boldsymbol{\Sigma}_{1.2}^{-1}[\bar{\boldsymbol{U}} - E_H\bar{\boldsymbol{U}} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\bar{\boldsymbol{T}} - E_H\bar{\boldsymbol{T}})], \quad (2.1)$$

where E_H denotes the expectation under H , and $\boldsymbol{X}' = (X_1,\ldots,X_n)$ and $\boldsymbol{Y}' = (Y_1,\ldots,Y_n)$.

Let now $\hat{\boldsymbol{\Sigma}}$, $\hat{E}_H(\bar{\boldsymbol{U}})$ and $\hat{E}_H(\bar{\boldsymbol{T}})$ denote the maximum likelihood estimators of $\boldsymbol{\Sigma}$, $E_H(\bar{\boldsymbol{U}})$ and $E_H(\bar{\boldsymbol{T}})$ respectively. It may be noted that these estimators are the functions of the maximum likelihood estimators of μ, ν, κ_1, and κ_2 where

$$\hat{\mu} = \bar{X}_0,\ \hat{\nu} = \bar{Y}_0,\ \hat{\kappa}_1 = A^{-1}(\bar{R}_1),\ \text{and}\ \hat{\kappa}_2 = A^{-1}(\bar{R}_2) \quad (2.2)$$

in the notation of Mardia (1972, Chapter 5). Then the resulting value of $g(\underset{\sim}{X},\underset{\sim}{Y})$ is

run on/

$$r^2_{XY} = [\underset{\sim}{\bar{U}} - \hat{E}_H(\underset{\sim}{\bar{U}}) - \hat{\underset{\sim}{\Sigma}}_{12}\hat{\Sigma}^{-1}_{22}(\underset{\sim}{\bar{T}} - \hat{E}_H\underset{\sim}{\bar{T}})]'\hat{\underset{\sim}{\Sigma}}^{-1}_{1\cdot 2}[\underset{\sim}{\bar{U}} - \hat{E}_H\underset{\sim}{\bar{U}} - \hat{\underset{\sim}{\Sigma}}_{12}\hat{\underset{\sim}{\Sigma}}^{-1}_{22}(\underset{\sim}{\bar{T}} - \hat{E}_H\underset{\sim}{\bar{T}})] \tag{2.3}$$

The rationale for (2.1) is obvious considering the conditional distribution of $\underset{\sim}{\bar{U}}|\underset{\sim}{\bar{T}}$ when $\underset{\sim}{\bar{U}}$ and $\underset{\sim}{\bar{T}}$ are jointly normal. Note that (2.2) and (2.3) are invariant under separate linear non-singular transformations on $\underset{\sim}{\bar{U}}$ and $\underset{\sim}{\bar{T}}$.

We now show that this procedure when applied to the bivariate case with

$$n\underset{\sim}{\bar{U}} = \sum_{i=1}^{n} X_iY_i \ , \quad n\underset{\sim}{\bar{T}}' = (\Sigma X_i, \Sigma X_i^2, \Sigma Y_i, \Sigma Y_i^2)$$

leads to r^2 where r is the ordinary correlation coefficient.

We assume that under H , X and Y are independent $N(\mu,\sigma_1^2)$ and $N(\mu,\sigma_2^2)$ respectively. Then it can be easily shown that $\underset{\sim}{\Sigma}_{1\cdot 2} = \sigma_1^2\sigma_2^2$, $\underset{\sim}{\Sigma}_{12}\underset{\sim}{\Sigma}^{-1}_{22} = (\nu,\mu,0,0)$ and (2.1) reduces to

$$\{\bar{U} - \mu\nu - \nu(\bar{X} - \mu) - \mu(\bar{Y} - \nu)\}^2/\sigma_1^2\sigma_2^2 , \tag{2.6}$$

where $\bar{X} = \sum_{i=1}^{n} X_i/n$ and $\bar{Y} = \sum_{i=1}^{n} Y_i/n$. Using the maximum likelihood estimators for μ, ν, σ_1^2 and σ_2^2 under H , (2.4) reduces to r^2 . Hence Cox's procedure does lead to a suitable correlation for the linear case.

3. The Correlation Coefficient

Without any loss of generality, let

$$\bar{X}_o = \bar{Y}_o = 0 \qquad (3.1)$$

Consequently, $\hat{\mu} = \hat{\nu} = 0$.

It can be shown that

run on/

$$\hat{\underset{\sim}{\Sigma}}_{11} = \frac{1}{4} \operatorname{diag}\{(1+\hat{\alpha}_2)(1+\hat{\alpha}_2^*) - 4\hat{\alpha}^2\hat{\alpha}^{*2} , (1+\hat{\alpha}_2)(1-\hat{\alpha}_2^*) , (1-\hat{\alpha}_2)(1-\hat{\alpha}_2^*), (1-\hat{\alpha}_2)(1+\hat{\alpha}_2^*)\} , \qquad (3.2)$$

$\hat{\underset{\sim}{\Sigma}}_{22} = \operatorname{diag}(a_1, a_2, a_1^*, a_2^*)$, and

$$\hat{\underset{\sim}{\Sigma}}_{12} = \begin{pmatrix} \hat{\alpha}^* a_1, & 0, & \hat{\alpha} a_1^*, & 0 \\ 0, & 0, & 0, & \hat{\alpha} a_2^* \\ 0, & 0, & 0, & 0 \\ 0, & \alpha^* a_2, & 0, & 0 \end{pmatrix} \qquad (3.3)$$

where

$$2a_1 = 1 + \hat{\alpha}_2 - 2\hat{\alpha}^2, \quad 2a_2 = 1 - \hat{\alpha}_2 \qquad (3.4)$$

$$2a_1^* = 1 + \hat{\alpha}_2^* - 2\hat{\alpha}^{*2}, \quad 2a_2^* = 1 - \hat{\alpha}_2^* \qquad (3.5)$$

$$\alpha = E \cos X, \quad \alpha^* = E \cos Y \qquad (3.6)$$

$$\alpha_2 = E \cos 2X, \quad \alpha_2^* = E \cos 2Y . \qquad (3.7)$$

On simplifying, we obtain

$$\hat{\underset{\sim}{\Sigma}}_{1\cdot 2} = \operatorname{diag}(a_1 a_1^*, a_1 a_2^*, a_2 a_2^*, a_2 a_1^*) \qquad (3.8)$$

Note that

$$\hat{\alpha} = \bar{R}_1, \; \hat{\alpha}_2 = 2\{1 - \bar{R}_1^2 - (\bar{R}_1/\hat{\kappa}_1)\} \tag{3.9}$$

$$\hat{\alpha}^* = \bar{R}_2, \; \hat{\alpha}_2^* = 2\{1 - \bar{R}_2^2 - (\bar{R}_2/\hat{\kappa}_2)\} \tag{3.10}$$

$$\hat{E}(\underset{\sim}{U}') = (\hat{\alpha}\hat{\alpha}^*, 0, 0, 0) \tag{3.11}$$

and

$$\hat{E}(\underset{\sim}{\bar{T}}') = (\hat{\alpha}, 0, \hat{\alpha}^*, 0) \; . \tag{3.12}$$

Substituting these results in (2.3), we obtain

$$r^2_{XY} = \left\{\frac{(\bar{U}_1 - \bar{R}_1\bar{R}_2)^2}{a_1 a_1^*} + \frac{\bar{U}_2^2}{a_1 a_2^*} + \frac{\bar{U}_3^2}{a_2 a_2^*} + \frac{\bar{U}_4^2}{a_1^* a_2}\right\} , \tag{3.13}$$

where $\bar{X}_o = \bar{Y}_o = 0$.

Note that

$$a_1 = \widehat{\text{var}}(\cos X) \; , \quad a_2 = \widehat{\text{var}}(\sin X)$$

run on/

$$a_1^* = \widehat{\text{var}}(\cos Y) \; , \quad a_2^* = \widehat{\text{var}}(\sin Y) \; .$$

Since $(\underset{\sim}{\bar{U}}', \underset{\sim}{\bar{T}}')$ is jointly normal for large n , $ng(\underset{\sim}{X}, \underset{\sim}{Y})$ is asymptotically χ_4^2 under H . Now since (2.1) and (2.2) are asymptotically equivalent, it follows that under H , nr^2_{XY} is asymptotically χ_4^2 .

4. Properties of the Correlation Coefficient.

(a) It can easily be seen that the population counterpart of (3.14) is

$$\rho_{XY}^2 = \rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2 \qquad (4.1)$$

where

$$\rho_1 = \text{corr}\{\cos(X-\mu_0),\ \cos(Y-\nu_0)\} ,$$

$$\rho_2 = \text{corr}\{\cos(X-\mu_0),\ \sin(Y-\nu_0)\} ,$$

$$\rho_3 = \text{corr}\{\sin(X-\mu_0),\ \cos(Y-\nu_0)\} ,$$

$$\rho_4 = \text{corr}\{\sin(X-\mu_0),\ \sin(Y-\nu_0)\} ,$$

and μ_0 and ν_0 are the mean directions of X and Y respectively (Note that corr(X,Y) denotes the ordinary correlation coefficient).

We have $$0 \le \rho_{XY}^2 \le 4 .$$

If X and Y are independent, we have $\rho_{xy}^2 = 0$. When $\rho_1^2 = \rho_2^2 = \rho_3^2 = \rho_4^2 = 1$, we have $\rho_{xy}^2 = 4$. In this case, $X = Y = 0$, with probability 1. For the perfect rotational dependence of the form $Y = (\pm X + \psi) \bmod 2\pi$, it is found that $\rho_{xy}^2 = 2$.

(b) The exact distribution of $\bar{\underset{\sim}{U}}|\bar{\underset{\sim}{T}}$ under (1.1) does not involve nuisance parameter (μ_1, κ_1) and (μ_2, κ_2) because $\bar{\underset{\sim}{T}}$ is sufficient for (μ_i, κ_i), $i = 1,2$. Also the exact distribution of r_{XY}^2 does not involve (μ_i, κ_i) , $i = 1,2$ by the same argument as in Cox (1975).

(c) Let $\underset{\sim}{x}$ and $\underset{\sim}{Y}$ take values on a p-dimensional torus, where $\underset{\sim}{X}'\underset{\sim}{X} = \underset{\sim}{Y}'\underset{\sim}{Y} = 1$. Then (4.11) immediately extends to

$$\rho_{XY}^2 = \sum_{i=1}^{p} \sum_{j=1}^{p} \{\mathrm{Corr}(X_i^*, Y_i^*)\}^2 ,$$

where $\underset{\sim}{X}^* = \underset{\sim}{AX}$, $\underset{\sim}{Y}^* = \underset{\sim}{BY}$, and, A and B are orthogonal matrices such that $E(\underset{\sim}{X}^*) = E(\underset{\sim}{Y}^*) = \underset{\sim}{e}_1$, where $\underset{\sim}{e}_1' = (1,0,0,\ldots,0)$. Its sample counterpart can be written, and the above discussion for $p = 2$ can be generalized.

5. Numerical Example

Downs (1974) considered the following data related to the estimated peak times (converted into angles) for two successive measurements of diastolic blood pressures.

θ = 30°, 15°, 11°, 4°, 348°, 347°, 341°, 333°, 332°, 28°

φ = 25°, 5°, 349°, 350°, 340°, 347°, 345°, 331°, 329°, 28°.

Then

$$r_1^2 = 0.974, \quad r_2^2 = 0.213, \quad r_3^2 = 0.152, \quad r_4^2 = 0.933 \quad ,$$

Hence $\quad r^2 = 2.27$,

run on/

which is large. Hence there is a strong evidence for dependence as it is expected on practical ground. Note that $nr^2 = 22.7$, and the 1% value of χ_4^2 is 13.3.

REFERENCES

COX, D.R. (1975). Discussion to Statistics of directional data, J. Roy. Statist. Soc. B, 37, 380-381.

DOWNS, T.D. (1974). Rotational angular correlations. In Biorythms and Human Reproduction. (M. Ferin et al., ed.), pp. 97-104, John Wiley, New York.

DOWNS, T.D. and EIFLER, C.W. (1975). Discussion to Statistics of directional data, J. Roy Statist. Soc. B, 37, 384.

MARDIA, K.V. (1972). Statistics of Directional Data. Academic Press, New York.

MARDIA, K.V. (1975). Statistics of Directional Data. J. Roy Statist. Soc. B, 37, 349-393.

PURI, M. L. and RAO. J. S. (1976). Problems of association for bivariate circular data and a new test of independence. Proc. Fourth Intern. Symp. Multivariate Analysis IV, (Ed. P. R. Krishnaiah). North Holland Publishing Company.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR - TR - 77 - 0447	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A ROBUST SPHERICAL CORRELATION COEFFICIENT AGAINST SCALE.		5. TYPE OF REPORT & PERIOD COVERED Interim rept.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Kanti V. Mardia ~~and~~ Madan L. Puri		8. CONTRACT OR GRANT NUMBER(s) AF- AFOSR ~~76~~-2927-76
9. PERFORMING ORGANIZATION NAME AND ADDRESS Indiana University Department of Mathematics Bloomington, IN 47401 11p.		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB, Washington, DC 20332		12. REPORT DATE 1977
		13. NUMBER OF PAGES 10
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

spherical correlation coefficient, von Mises distribution, robustness, invariance, maximum likelihood estimator

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

There are several bi-directional situations where it is required to obtain a measure of correlation. The question has been raised often that the bi-directional correlation coefficients so far known are not scale invariant even asymptotically. Following Cox's procedure (Cox, 1975, J. Roy. Statist. Soc. B, 380-381), we introduce a new correlation coefficient, which has the desirable property of being invariant under (Continued on reverse side)

it is introduced

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE(When Data Entered)

20. Abstract

scale for large samples with von Mises marginals. We obtain its asymptotic distribution under the hypothesis of independence. We examine its properties, and give a numerical example.

is obtained and examined.

UNCLASSIFIED